DELOCALIZED EQUIVARIANT COHOMOLOGY OF SYMMETRIC PRODUCTS

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Abstract. For any closed complex manifold X, we calculate the Poincaré and Hodge polynomials of the delocalized equivariant cohomology $H^*(X^n, S_n)$ with a grading specified by physicists. As a consequence, we recover a special case of a formula for the elliptic genera of symmetric products in Dijkgraaf-Moore-Verlinde-Verlinde [8]. For a projective surface X, our results matches with the corresponding formulas for the Hilbert scheme of $X^{[n]}$. We also give geometric construction of an action of a Heisenberg superalgebra on $\sum_{n\geq 0} H^{*,*}(X^n, S_n)$, imitating the constructions for equivariant K-theory by Segal [25] and Wang [27]. There is a corresponding version for $H^{-*,*}$.

1. Introduction

Given a smooth manifold M and a finite group G of diffeomorphisms, one can study the orbifold M/G by its de Rham cohomology $H^*(M/G)$ following Satake [24]. It is easy to see that

$$H^*(M/G) \cong H^*(M)^G. \tag{1}$$

In particular, given a manifold X, the permutation group S_n acts on the n-fold Cartesian product X^n by permuting the factors. The orbifold $X^{(n)} = X^n/S_n$ is called the *n-th symmetric product* of X. From (1), one obtains

$$H^*(X^{(n)}) \cong (H^*(X)^{\otimes n})^{S_n} = S^n(H^*(X)), \tag{2}$$

where $S^n(H^*(X))$ is the n-th graded symmetric product of $H^*(X)$. From this Macdonald [20] obtained the following formula:

$$\sum_{n\geq 0} P_t(X^{(n)})q^n = \frac{\prod_{d\ odd}(1+t^dq)^{b_d(X)}}{\prod_{d\ even}(1-t^dq)^{b_d(X)}}, \tag{3}$$

where $P_t(X^{(n)})$ is the Poincaré polynomial of $X^{(n)}$, and $X^{(0)}$ is a point. In particular, taking $t=-1$, one obtains the the generating functional for the Euler numbers:

$$\sum_{n\geq 0} \chi(X^{(n)})q^n = \frac{1}{(1-q)^{\chi(X)}}. \tag{4}$$

An alternative proof of (4) was given in Zagier [28]. It exploits the following idea: given any orbifold M/G, for $g \in G$, consider the Lefschetz number $L_g = tr\,(g_*|_{H^{even}(M)}) - tr\,(g_*|_{H^{odd}(M)})$. By standard character theory,

$$\chi(M/G) = \chi(H^*(M)^G) = \frac{1}{|G|}\sum_{g\in G} L_g = \sum_{[g]\in G_*} \frac{1}{|Z_g|} L_g,$$

where $|G|$ is the number of elements in G, Z_g is the centralizer of g, and G_* denotes the set of all conjugacy classes of G. Applying equivariant Atiyah-Singer theorem to the de Rham complex, one gets $L_g = \chi(M^g)$, where $M^g = \{x \in M : g(x) = x\}$. Hence

$$\chi(M/G) = \frac{1}{|G|}\sum_{g\in G}\chi(M^g) = \sum_{[g]}\frac{1}{|Z_g|}\chi(M^g). \tag{5}$$

Such an approach was used by Hirzebruch to compute the signatures of symmetric products. See Zagier [28], §9 for details.

Unfortunately $H^*(M/G)$ is not always sufficient for studies on orbifolds. For example, it is not suitable for the equivariant K-theory. To generalize the isomorphism between the K-theory and cohomology on compact spaces given by the Chern character, Baum and Connes [6] defined the delocalized equivariant cohomology

$$H^*(M,G) = (\bigoplus_{g\in G} H^*(M^g))^G$$

and an equivariant Chern character $ch_G : K_G^*(M) \to H^*(M,G)$ which they showed to be an isomorphism. It turns out that $H^*(M,G)$ is also the correct cohomology for the string theory on orbifolds studied by Dixon, Harvey, Vafa and Witten [9, 10] (see also Vafa-Witten [26], §4.1). One easily sees that

$$H^*(M,G) \cong \bigoplus_{[g]\in G_*} H^*(M^g)^{Z_g}.$$

The components $H^*(M^g)^{Z_g}$ for nontrivial conjugacy classes correspond to the twisted sectors. The connection between the orbifold string theory and the delocalized equivariant cohomology was made by Atiyah and Segal [2]. They interpreted the following formula for orbifold Euler number which appeared in [9] as the Euler number of $H^*(M,G)$ or equivalently $K_G^*(M)$:

$$\chi(M,G) = \frac{1}{|G|}{\sum_{g,h}}' \chi(M^{\langle g,h\rangle}) = \sum_{[g]\in G_*}\frac{1}{|Z_g|}\sum_{h\in Z_g}\chi(M^{\langle g,h\rangle}), \tag{6}$$

where $\langle g,h\rangle$ is the group generated by g and h, the first sum is taken over commutating pairs $(g,h) \in G\times G$. From (5) and (6), one obtains (cf. Hirzebruch and Höfer [17])

$$\chi(M,G) = \sum_{[g]\in G_*}\chi(M^g/Z_g). \tag{7}$$

See also Roan [23] for a mathematical expositions of the orbifold Euler number. In a more recent paper by Vafa and Witten [26], the following formula corresponding to (4) was proved:

$$\sum_{n\geq 0}\chi(X^n,S_n)q^n = \prod_{l\geq 1}\frac{1}{(1-q^l)^{\chi(X)}}. \tag{8}$$

The proof in [26] is in the spirit of Macdonald [20] mentioned above. A proof using Lefschetz numbers has been given by Hirzebruch and Höfer [17].

Vafa and Witten noticed that $\sum q^n H^*(X^n, S_n)$ is the Fock space of the Heisenberg superalgebra generated by $H^*(X)$. Motivated by this result, Nakajima [22] and Grojnowski [16] independently obtained the geometric construction of the representation on $\sum q^n H_*(X^{[n]})$, where $X^{[n]}$ is n-th Hilbert scheme of points of a

surface X. Partly motivated by a footnote in [16], Segal [25] outlined the similar constructions for the equivariant K-theory $K^*_{S_n}(X^n)$. Wang [27] generalized Segal's constructions to the case of $K^*_{G_n}(X^n)$, where G is a finite group, X is a G-space, and G_n the wreath product of S_n with G^n. This leads to constructions of vertex representations via finite groups (see Frenkel-Jing-Wang [11, 12]).

Under the isomorphism with $K^*_G(M)$ given by the equivariant Chern character, $H^*(M,G)$ acquires a $\mathbb{Z}_2$-grading. In the physics literature, a rule of assigning a grading possibly by fractional numbers has been well-known. For a complex orbifold M/G, a number F_g can be defined for each component of M^g, $g \in G$. Set

$$H^{p,q}(M,G) = \bigoplus_{p,q \geq 0} H^{p-F_g,q-F_g}(M^g/Z_g).$$

See e.g. Zaslow [29] and also §2. In this paper, we show that for a complex manifold X, F_g is just half of codimension of $(X^n)^g$ for all $g \in S_n$. One can define Poincaré and Hodge polynomials of $H^*(X^n, S_n)$ with the above grading. We compute such polynomials in this paper. For algebraic surfaces, our formulas coincide with the results in Göttsche [13], Göttsche-Soergel[15] and Cheah [7]. We also follow the constructions in Segal [25] and Wang [27] to construct the representation of the Heisenberg superalgebra on $\sum H^*(X^n, S_n)$. We use the Poincaré duality in our construction.

Using our formula for the Hodge polynomials, we give a mathematical proof of a special case of a formula for the elliptic genera of symmetric products of complex manifolds found by Dijkgraaf, Moore, E. Verlinde and H. Verlinde by physical arguments.

The same methods work for $H^{-*,*}(X^n, S_n)$. Also most of our discussions can be carried out for wreath products.

2. Preliminaries on the delocalized equivariant cohomology

Let G be a finite group acting on a manifold M, Baum and Connes [6] defined the *delocalized cohomology* as follows: let $\widehat{M}$ be the disjoint union of M^g, $g \in G$, then there is a natural action of G on $\widehat{M}$, set

$$H^*(M,G) = H^*(\widehat{M})^G = (\oplus_{g\in G} H^*(M^g))^G, \tag{9}$$

$$H^*_c(M,G) = H^*_c(\widehat{M})^G = (\oplus_{g\in G} H^*_c(M^g))^G. \tag{10}$$

Breaking into conjugacy classes, it is clear that

$$H^*(M,G) = \oplus_{g\in G_*} H^*(M^g)^{Z_g}, \quad H^*_c(M,G) = \oplus_{g\in G_*} H^*_c(M^g)^{Z_g}.$$

Assume that G_1 and G_2 are two finite groups, and $\phi : G_1 \to G_2$ is a group homomorphism. A map f between a G_1-space M_1 and a G_2-space M_2 is called *equivariant* (with respect to ϕ) if $f(g \cdot x) = \phi(g) \cdot f(x)$ for any $g \in G_1$, $x \in M_1$. Such a map induces an equivariant map $\hat{f} : \widehat{M_1} \to \widehat{M_2}$, and hence a homomorphism $f^* : H^*(M_2, G_2) \to H^*(M_1, G_1)$ and $f^*_c : H^*_c(M_2, G_2) \to H^*_c(M_1, G_1)$. It is then easy to see that the delocalized cohomology is functorial.

If M_1 is a G_1-space and M_2 a G_2-space, then $M_1 \times M_2$ is naturally a $G_1 \times G_2$-space. Furthermore, if $g_1 \in G_1$ and $g_2 \in G_2$, then $((M_1 \times M_2)^{(g_1,g_2)} = M_1^{g_1} \times M_2^{g_2}$, and $Z_{(g_1,g_2)}(G_1 \times G_2) = Z_{g_1}(G_1) \times Z_{g_2}(G_2)$. By the ordinary Künneth theorem,

one obtains isomorphisms

$$\kappa : H^*(M_1 \times M_2, G_1 \times G_2) \cong H^*(M_1, G_1) \otimes H^*(M_2, G_2),$$
$$\kappa_c : H_c^*(M_1 \times M_2, G_1 \times G_2) \cong H_c^*(M_1, G_1) \otimes H_c^*(M_2, G_2).$$

Let M be a G-space and G' is a subgroup of G, we now define some functors which we will use later. First of all, the identity map on M is equivariant with respect to the inclusion $G' \to G$. We denote the induced homomorphism $H^*(M, G) \to H^*(M, G')$ by $\mathrm{Res}_{G'}^G$, or simply Res when there is no confusion. With respect to the isomorphisms

$$H^*(M, G) \cong \oplus_{[g] \in G_*} H^*(M^g)^{Z_g(G)}, \quad H^*(M, G') \cong \oplus_{[g'] \in G'_*} H^*(M^{g'})^{Z_{g'}(G')},$$

we can give Res explicitly: if $g \in G$ is not conjugate by elements in G to any element in G', then $\mathrm{Res}|_{H^*(M^g)^{Z_g(G)}} = 0$; otherwise, assume that g is conjugate by elements in G to $g_1', \cdots, g_k' \in G'$ which have mutually different conjugacy classes in G', then $H^*(M^g)^{Z_g(G)} \cong H^*(M^{g_i'})^{Z_{g_i'}(G)}$ for $i = 1, \cdots, k$ and $\mathrm{Res}|_{H^*(M^g)^{Z_g(G)}}$ is given by the direct sum of the inclusions $H^*(M^{g_i'})^{Z_{g_i'}(G)} \hookrightarrow H^*(M^{g_i'})^{Z_{g_i'}(G')}$. We define another homomorphism $\mathrm{Ind}_{G'}^G : H^*(M, G') \to H^*(M, G)$ as follows: for $\alpha \in H^*(M^g)^{Z_g(G')}$, $g \in G'$,

$$\mathrm{Ind}_{G'}^G(\alpha) = \frac{1}{|Z_g(G)|} \sum_{h \in Z_g(G)} h^*(\alpha) \in H^*(M^g)^{Z_g(G)} \hookrightarrow H^*(M, G).$$

We now consider a $\mathbb{Z}$-graing of $H^*(M, G)$ suggested by physicists. Let M be a complex manifold and G acts by biholomorphic maps. Denote by N^g the normal bundle of a component M_i^g of M^g in M. Then there is a natural decomposition $N^g = \oplus_j N^g(\theta_j)$, where $N^g(\theta_j)$ is a complex subbundle on which g acts as $e^{\sqrt{-1}\theta_j}$ where $0 < \theta_j < 2\pi$ (these angles will be called action angles). Zaslow [29] suggested the shift of the bigrading on $H^{*,*}(M_i^g)$ by (F_g, F_g), and hence the shift in grading on $H^*(M_i^g)$ by $2F_g$, where

$$F_g = \frac{\sum_j \mathrm{rank}_{\mathbb{C}} N^g(\theta_j) \cdot \theta_j}{2\pi}. \tag{11}$$

It is interesting to notice that here we are using some data which appeared in equivariant index theory (Atiyah-Singer [3]). Notice that F_g may not be always an integer, but there are conditions to ensure it is.

Example 2.1. Let X be a complex manifold, then $\mathbb{Z}_n$ acts on X^n by cyclic permutions of the factors. Denote by σ the generator of $\mathbb{Z}_n$, then $(X^n)^\sigma = \Delta_n(X)$, the diagonal. Since the cyclic matrix has eigenvalues $e^{2\pi i j/n}$, one can easily deduce that

$$N^\sigma = \oplus_{j=1}^{n-1} N^\sigma(2\pi j/n),$$

where each $N^\sigma(2\pi j/n)$ is isomorphic to the complex tangent bundle of $\Delta_n(X)$. Hence by (11)

$$F_\sigma = \dim_{\mathbb{C}} X \cdot \sum_{k=1}^{n-1} k/n = \dim_{\mathbb{C}} X \cdot (n-1)/2.$$

Note that this is exactly the half of the complex codimension of the diagonal in X^n. For this to be an integer for all n, one only needs to assume that M has even complex dimension.

Example 2.2. Let M/G be a complex orbifold of dimension n. Zaslow [29] showed that $F_{g^{-1}} = \operatorname{codim} M^g - F_g$ by the following observation: $M^g = M^{g^{-1}}$, and if the action angles on the normal bundle of M^g are $\{\theta_i\}$, then those of g^{-1} are $\{2\pi - \theta_i\}$. Now if g is conjugate to g^{-1}, then $F_g = \frac{1}{2}\operatorname{codim} M^g$. Since every element in S_n is conjugate to its inverse, this gives an alternative calculation for Example 2.1.

Example 2.3. If M is a Calabi-Yau manifold and G is a finite automorphism group which preserves the holomorphic volume form, then F_g is an integer for all $g \in G$. Indeed, g acts in the fiber of the normal bundle by a matrix of determinant 1, while the determinant can be computed as $\exp(2\pi\sqrt{-1}F_g)$.

3. Preliminaries on the symmetric products of graded vector spaces

For a $\mathbb{Z}$-graded finite dimensional vector space (over a field $\mathbf{k} \cong \mathbb{R}$ or $\mathbb{C}$) $V = \sum_{d\in\mathbb{Z}} V_d$ such that $b_d(V) = \dim V_d < \infty$ for all d and $V_d = 0$ for $d < 0$, the Poincaré series of V is by definition

$$p_t(V) = \sum_{d\geq 0} b_d(V)t^d.$$

Denote by $\mathcal{GV}_{\geq}$ the set of all such graded vector spaces. Then $\mathcal{GV}_{\geq}$ admits several operations. For any integer m and any graded vector space V, $V[m]$ is the graded vector space with $V[m]_d = V_{d-m}$. For positive integer m and $V \in \mathcal{GV}_{\geq}$, $V[m] \in \mathcal{GV}_{\geq}$. For $V, V' \in \mathcal{GV}_{\geq}$, let $V \oplus V'$ be the graded vector space with $(V \oplus V')_d = V_d \oplus V'_d$. Then $V \oplus V' \in \mathcal{GV}_{\geq}$. Alos let $V \otimes V'$ be the graded vector space with $(V \otimes V')_d = \oplus_{p+q=d} V_p \otimes V'_q$. Then $V \otimes V' \in \mathcal{GV}_{\geq}$. Clearly we have

$$\begin{aligned}
&p_t(V[m]) = t^m p_t(V),\\
&p_t(V \oplus V') = p_t(V) + p_t(V'),\\
&p_t(V \otimes V') = p_t(V) \cdot p_t(V').
\end{aligned}$$

Since $\mathcal{GV}_{\geq}$ is an abelian semigroup under $\oplus$, one can consider its Grothendieck group $GK_{\geq}$ (cf. Atiyah [1], §2.1). Then p_t is a ring homomorphism $p_t : GK_{\geq} \to \mathbb{Z}[[t]]$. Denote by $\mathcal{V}$ the space of (ungraded) finite dimensional vector spaces. This is also an abelian semigroup, so one can take its Grothendieck group K. This is just the K-theory of a point. We regard $GK_{\geq}$ as $K[[t]]$. There is a map $D : \mathcal{V} \to \mathbb{Z}$ given by taking the dimensions of the vector spaces. It extends to $D : K \to \mathbb{Z}$. If we regard $GK_{\geq}$ as $K[[t]]$, then p_t is nothing but the extension of D to $K[[t]]$.

Now we recall some power operation in K-theory (Atiyah [1], §3.1). Given $V \in \mathcal{V}$, let $S^n(V)$ be the n-symmetric product of V. Set $S_t^*(V) = \sum_{n\geq 0} t^n S^n(V)$. This extends to a map $S_t^* : K \to K[[t]]$. We also regrad this as giving a map $S^* : \mathcal{V} \to GK_{\geq}$: for $V \in \mathcal{V}$, elements of $S^n(V)$ has degree n. Then $D(S_t^*(V)) = p_t(S^*(V))$. Since

$$S^n(V \oplus V') \cong \oplus_{p+q=n} S^p(V) \otimes S^q(V'),$$

for $V, V' \in \mathcal{V}$, we get

$$S_t^*(V \oplus V') = S_t^*(V) S_t^*(V').$$

For a one dimensional vector space L, we have

$$S_t^*(L) = 1 + tL + t^2 L^{\otimes 2} + \cdots,$$

hence

$$D(S_t^*(L)) = 1 + t + t^2 + \cdots = \frac{1}{1-t}.$$

Hence by writing $V \in \mathcal{V}$ as a direct sum of one-dimensional subspaces, one has

$$p_t(S^*(V)) = D(S^*_t(V)) = \frac{1}{(1-t)^{\dim V}}.$$

One can also consider the anti-symmetric produces: $\Lambda^*_t : K \to K[[t]]$ and $\Lambda^* : K \to \mathcal{GV}_{\geq}$ given by $\Lambda^*_t(V) = \sum_{n\geq 0} t^n \Lambda^n(V)$ and $\Lambda^*(V) = \sum_{n \geq 0} \Lambda^n(V)$ for $V \in \mathcal{V}$, where elements in $\Lambda^n(V)$ are given degree n. Then $D(\Lambda^*_t(V)) = p_t(\Lambda^*(V))$. Since

$$\Lambda^n(V \oplus V') \cong \oplus_{p+q=n} \Lambda^p(V) \otimes \Lambda^q(V'),$$

for $V, V' \in \mathcal{V}$, we get

$$\Lambda^*_t(V \oplus V') = \Lambda^*_t(V)\Lambda^*_t(V').$$

For a one dimensional vector space L, we have

$$\Lambda^*_t(L) = 1 + tL,$$

hence

$$D(\Lambda^*_t(L)) = 1 + t.$$

Therefore for $V \in \mathcal{V}$, one has

$$p_t(\Lambda^*(V)) = D(\Lambda^*_t(V)) = (1+t)^{\dim V}.$$

Similarly, for a bi-graded vector space $W = \sum_{m,n\in\mathbb{Z}} W_{mn}$ such that $h_{m,n}(W) = \dim W_{mn} < \infty$ for all $m, n \in \mathbb{Z}$ and $V_{mn} = 0$ if either $m < 0$ or $n < 0$, define the Hodge series of W by

$$h_{x,y}(W) = \sum_{m,n\geq 0} h_{m,n}(W) x^m y^n.$$

We denote by $2\mathcal{GV}_{\geq}$ the space of all such bi-graded vector spaces. It is routine to define $W[l,m], W \oplus W', W \otimes W' \in 2\mathcal{GV}_{\geq}$ for $W, W' \in 2\mathcal{GV}$, $l, m \geq 0$. Furthermore, we clearly have

$$\begin{aligned} h_{x,y}(W[l,m]) &= x^l y^m h_{x,y}(W), \\ h_{x,y}(W \oplus W') &= h_{x,y}(W) + h_{x,y}(W'), \\ h_{x,y}(W \otimes W') &= h_{x,y}(W) \cdot h_{x,y}(W'). \end{aligned}$$

Denote by $2GK_{\geq}$ the Grothendieck group of the abelian semigroup $2cGV_{\geq}$. Then $h_{x,y}$ extends to a homomorphism $h_{x,y} : 2GK \to \mathbb{Z}[[x,y]]$. We can identify $2GK$ as $GK[[y]]$ or $K[[x,y]]$. With respect to the former identification, $h_{x,y}$ is identified with the extension of P_x. With respect to the latter, $h_{x,y}$ is identified with the extension of dim.

Given $V \in \mathcal{GV}$, let $T^*(V)$ be the tensor algebra of V, I the ideal of $T^*(V)$ generated by elements of the form $v \otimes w - (-1)^{pq} w \otimes v$, where $v \in V_p$, $w \in V_q$. Set $S^*(V) = T^*(V)/I$. There is a natural decomposition $S^*(V) = \oplus_{n \geq 0} S^n(V)$, where $S^n(V)$ is called the *n-th graded symmetric product of V*. We regard $S^*(V)$ as a $\mathbb{Z}\times\mathbb{Z}$-graded vector space. For $v_1 \in V_{|v_1|}, \cdots, v_n \in V_{|v_n|}$, denote by $v_1 \odot v_2 \odot \cdots \odot v_m$ the image of $v_1 \otimes \cdots \otimes v_n$ in $S^n(V)$. Then elements of such form generate $S^n(V)$, we give such an element a bi-degree $(|v_1| + \cdots + |v_n|, n)$. With this bi-grading, we get a map $S^* : GK \to 2GK$ by $S^*(V) = \sum_{n\geq 0} S^n(V)$. Alternatively, we get a map $S^*_y : GK \to GK[[y]]$ by $S^*_y(V) = \sum_{n \geq 0} y^n S^n(V)$. We clearly have

$$\sum_{n\geq 0} h_{t,q}(S^n(V)) = \sum_{n \geq 0} p_t(S^n(V)) q^n.$$

In other words, $h_{t,q}(S^*(V)) = p_t(S^*_q(V))$. Since

$$S^n(V \oplus V') \cong \oplus_{p+q=n} S^p(V) \otimes S^q(V'),$$

for $V, V' \in \mathcal{GV}$, we get

$$S^*_t(V \oplus V') = S^*_t(V) S^*_t(V').$$

For a one-dimensional graded vector space L which concentrates at degree d, when d is even we have

$$S^*_q(L) = 1 + qL + q^2 L^{\otimes 2} + \cdots ,$$

hence

$$p_t(S^*_q(L)) = 1 + qt^d + q^2 t^{2d} + \cdots = \frac{1}{1 - t^d q}.$$

When d is odd, we have

$$S^*_q(L) = 1 + qL,$$

hence

$$p_t(S^*_q(L)) = 1 + t^d q.$$

Hence by writing $V \in \mathcal{GV}$ as a direct sum of one-dimensional graded subspaces, one obtains the following well-known formula

$$\sum_{n \geq 0} p_t(S^n(V)) q^n = \frac{\prod_{d\ odd} (1 + t^d q)^{b_d(V)}}{\prod_{d\ even} (1 - t^d q)^{b_d(V)}}.$$

As a corollary, one gets

$$\sum_{n \geq 0} p_t(S^n(V[m])) q^n = \frac{\prod_{d\ odd} (1 + t^d q)^{b_{d-m}(V)}}{\prod_{d\ even} (1 - t^d q)^{b_{d-m}(V)}}.$$

Let $W = \oplus_{p,q \in \frac{1}{2}\mathbb{Z}} W_{pq}$ be a $\frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$-graded vector space such that $W^{pq} \neq \{0\}$ only if $p + q \in \mathbb{Z}$. We regard W as a $\mathbb{Z}$-graded vector space: elements in $W^{p,q}$ is given the degree $p + q$. Then the n-th graded symmetric symmetric product $S^n(W)$ can be defined. It has a natural $\frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$-grading: for elements $w_1 \in W_{p_1 q_1}, \cdots, w_n \in W_{p_n q_n}$, $w_1 \odot \cdots \odot w_n$ has degree $(p_1 + \cdots + p_n, q_1 + \cdots + q_n)$. Assume that $h_{p,q}(W) = \dim W_{pq} < \infty$ for all $p, q \in \frac{1}{2}\mathbb{Z}$, set

$$h_{x,y}(W) = \sum_{r,s \in \frac{1}{2}\mathbb{Z}} h_{r,s}(W) x^r y^s.$$

The above discussions can be repeated to obtain the following formula:

$$\sum_{n \geq 0} h_{x,y}(S^n(W)) q^n = \frac{\prod_{s+t\ odd} (1 + x^s y^t q)^{h_{s,t}(W)}}{\prod_{s+t\ even} (1 - x^s y^t q)^{h_{s,t}(W)}}. \tag{12}$$

As a corollary, one gets for $l, m \in \frac{1}{2}\mathbb{Z}$ with $l + m \in \mathbb{Z}$,

$$\sum_{n \geq 0} h_{x,y}(S^n(W[l,m])) q^n = \frac{\prod_{s+t\ odd} (1 + x^s y^t q)^{h_{s-l,t-m}(V)}}{\prod_{s+t\ even} (1 - x^s y^t q)^{b_{s-l,t-m}(V)}}. \tag{13}$$

4. The orbifold Poincaré polynomials of the symmetric product

Recall that any element of S_n can be uniquely written as a products of disjoint cycles. Denote by $N_l(g)$ the number of l-cycles in g. The sequence $N(g) = (N_1(g), N_2(g), \cdots,)$ is called the *cycle type* of g. Each cycle type corresponds to a unique conjugacy class. Given any element $g \in S_n$ of type $N = (N_1, N_2, \cdots)$, there is an isomorphism

$$Z_g \cong S_{N_1} \times (S_{N_2} \ltimes \mathbb{Z}_2^{N_2}) \times \cdots \times (S_{N_n} \ltimes \mathbb{Z}_n^{N_n}),$$

where each S_{N_l} is given by permutating the l-cycles of g, and each $\mathbb{Z}_l$ is to the cyclic group generated by an l-cyle in g. We have

$$|Z_g| = \prod_{l=1}^n N_l! l^{N_l}.$$

Denote by $\Delta_l(X)$ the diagonal in X^l. It is clear that

$$(X^n)^g = \prod_{l=1}^n \Delta_l(X)^{N_l} \cong \prod_{l=1}^n X^{N_l}, \tag{14}$$

where each cycle of g contributes a factor of X. Furthermore, the action of Z_g can be explicitly described as follows: each copy of $\mathbb{Z}_l$ acts trivially, each copy of S_{N_l} permutes the N_l copies of $\Delta_l(X)$. Hence we have

$$(X^n)^g / Z_g \cong \prod_{l=1}^n X^{N_l}/S_{N_l} = \prod_{l=1}^n X^{(N_l)}. \tag{15}$$

Therefore, from (5), one gets

$$\begin{aligned}
\sum_{n\geq 0} \chi(X^{(n)}) q^n &= \sum_{n \geq 0} q^n \sum_{\sum lN_l = n} \frac{1}{\prod_{l=1}^n N_l! l^{N_l}} \chi(\prod_{l=1}^n X^{N_l}) \\
&= \sum_{n\geq 0} \sum_{\sum lN_l = n} \prod_{l=1}^n \frac{1}{N_l!} (\frac{\chi(X) q^l}{l})^{N_l} = \prod_{l=1}^n \sum_{N_l \geq 0} \frac{1}{N_l!} (\frac{\chi(X) q^l}{l})^{N_l} \\
&= \prod_{l=1}^n \exp\left(\frac{\chi(X) q^l}{l}\right) = \exp\left(\sum_{l=1}^n \frac{\chi(X) q^l}{l}\right) \\
&= \exp(-\chi(X)\log(1-q)) = \frac{1}{(1-q)^{\chi(X)}}.
\end{aligned}$$

This is the proof of (4) in Zagier [28], §9. Using (5) for M^g with the action of Z_g, Hirzebruch and Höfer [17] showed that

$$\chi(M, G) = \sum_{[g] \in G} \chi(M^g / Z_g). \tag{16}$$

Combining this expression with (15) and using (4), one gets

$$\begin{aligned}
& \sum_{n\geq 0} \chi(X^n, S_n) q^n \\
= & \sum_{n\geq 0} q^n \sum_{\sum lN_l = n} \chi(\prod_{l=1}^{n} X^{(N_l)}) = \sum_{n\geq 0} \sum_{\sum lN_l=n} \prod_{l=1}^{n} (\chi(X^{(N_l)}) q^{lN_l}) \\
= & \prod_{l\geq 1} \sum_{N_l\geq 0} \chi(X^{(N_l)}) q^{lN_l} = \prod_{l\geq 1} \frac{1}{(1-q^l)^{\chi(X)}}.
\end{aligned}$$

This calculation was given by Hirzebruch and Höfer [17] for surfaces. It clearly works in general.

Now we come to the calculations for Poincaré polynomials. Using the identification (15), one sees that

$$H^*(X^n, S_n) \cong \bigoplus_{\sum lN_l = n} H^*(\prod_{l=1}^{n} X^{(N_l)}) = \bigoplus_{\sum lN_l=n} \bigotimes_{N_l \geq 1} H^*(X^{(N_l)}).$$

By the isomorphism (1), we have

$$H^*(X^n, S_n) \cong \bigoplus_{\sum lN_l=n} \bigotimes_{l=1}^{n} S^{N_l}(H^*(X)),$$

as vector spaces. This is what Vafa and Witten [26] used to prove (8). We now analyze the grading shifts of the twisted sectors. This can be reduced to the case of the action of an n-cycle σ on X^n. We assume that X is a complex manifold for the time being. By Example 2.1, the shift is exactly the half of the real codimension of the diagonal. This suggests that for any manifold X of dimension $2m$, not necessarily complex, each $H^*(M^g)^{Z_g}$ should be shifted by half of the codimension of $(X^n)^g$ in X^n. I.e., as $\mathbb{Z}$-graded vector spaces,

$$H^*(X^n, S_n) \cong \bigoplus_{\sum lN_l=n} \bigotimes_{l=1}^{n} S^{N_l}(H^*(X)[m(l-1)]). \tag{17}$$

Then we have

$$\begin{aligned}
& \sum_{n\geq 0} P_t(X^n, S_n) q^n = \sum_{n\geq 0} q^n \sum_{\sum lN_l=n} P_t(\prod_{l=1}^{n} S^{N_l}(H^*(X)[m(l-1)])) \\
= & \sum_{n\geq 0} q^n \sum_{\sum lN_l=n} \prod_{l\geq 1} P_t(S^{N_l}(H^*(X)[m(l-1)])) \\
= & \prod_{l\geq 1} \sum_{N_l\geq 0} P_t(S^{N_l}(H^*(X)[m(l-1)])) q^{lN_l} = \prod_{l\geq 1} \frac{\prod_{d\ odd}(1+t^d q^l)^{b_{d-m(l-1)}(X)}}{\prod_{d\ even}(1-t^d q^l)^{b_{d-m(l-1)}(X)}}.
\end{aligned}$$

When m is even, we also have

$$\sum_{n\geq 0} P_t(X^n, S_n) q^n = \prod_{l\geq 1} \frac{\prod_{d\ odd}(1+t^{m(l-1)+d} q^l)^{b_d(X)}}{\prod_{d\ even}(1-t^{m(l-1)+d} q^l)^{b_d(X)}}. \tag{18}$$

To summarize, we have proved the following

Theorem 4.1. *For a $2m$-dimensional manifold X, if $H^*(X^n, S_n)$ is graded as in (17), then the generating functional of the Poincaré polynomials of $H^*(X^n, S_n)$ is given by:*

$$\sum_{n\geq 0} P_t(X^n, S_n)q^n = \prod_{l\geq 1} \frac{\prod_{d\ odd}(1+t^d q^l)^{b_{d-m(l-1)}(X)}}{\prod_{d\ even}(1-t^d q^l)^{b_{d-m(l-1)}(X)}}. \tag{19}$$

5. Hodge polynomials of the symmetric products

Let M be a complex manifold, G a finite group of bi-holomorphic transformations. Then M/G is a complex V-manifold, hence one can consider the Dolbeault cohomology of G-invariant forms:

$$H^{*,*}(M/G) = H^{*,*}(M)^G.$$

Now we assume that X is a complex manifold of complex dimension m. Then the natural action of S_n on X^n is bi-holomorphic. We then have

$$H^{*,*}(X^{(n)}) \cong S^n(H^{*,*}(X)).$$

By results from §3, we then have the following

Theorem 5.1. *For a compact complex manifold X, we have*

$$\sum_{n\geq 0} h_{x,y}(X^{(n)})q^n = \frac{\prod_{s+t\ odd}(1+x^s y^t q)^{h^{s,t}(X)}}{\prod_{s+t\ even}(1-x^s y^t q)^{h^{s,t}(X)}}. \tag{20}$$

Recall Hirzebruch's χ_y-genus for a compact complex manifold M is

$$\chi_y(M) = \sum_{p,q\geq 0} (-1)^q h^{p,q}(M) y^p.$$

I.e. $\chi_y(M) = h_{y,-1}(M)$. It is well-known that $\chi_{-1}(M) = \chi(M)$, $\chi_1(M) = \mathrm{sign}(M)$ and $\chi_0(M) = p_a(M)$, the arithmetic genus of M in Hirzebruch's sense.

Corollary 5.1. *For a compact complex manifold X we have*

$$\sum_{n\geq 0} \chi_{-y}(X^{(n)})q^n = \exp\left(\sum_{m\geq 1} \frac{\chi_{-y^m}(X)}{m} q^m\right), \tag{21}$$

$$\sum_{n\geq 0} p_a(X^{(n)})q^n = \frac{1}{(1-q)^{p_a(X)}}, \tag{22}$$

$$\sum_{n\geq 0} \mathrm{sign}(X^{(n)})q^n = \frac{1}{(1-q^2)^{\chi(X)/2}} \left(\frac{1+q}{1-q}\right)^{\mathrm{sign}(X)/2}. \tag{23}$$

Proof. From (20), it is easy to see that

$$
\begin{aligned}
&\sum_{n\geq 0}\chi_{-y}(X^{(n)})q^n = \frac{\prod_{s+t\ odd}(1+(-y)^s(-1)^t q)^{h_{s,t}(X)}}{\prod_{s+t\ even}(1-(-y)^s(-1)^t q)^{h_{s,t}(X)}} \\
= \quad & \frac{\prod_{s+t\ odd}(1-y^s q)^{h_{s,t}(X)}}{\prod_{s+t\ even}(1-y^s q)^{h_{s,t}(X)}} = \exp\left(-\sum_{s,t\geq 0}(-1)^{s+t}h^{s,t}(X)\log(1-y^s q)\right) \\
= \quad & \exp\left(\sum_{s,t\geq 0}(-1)^{s+t}h^{s,t}(X)\sum_{m>1}\frac{1}{m}(y^s q)^m\right) \\
= \quad & \exp\left(\sum_{m>1}\frac{1}{m}\sum_{s,t\geq 0}(-1)^{s+t}h^{s,t}(X)y^{ms}q^m\right) = \exp\left(\sum_{m>1}\frac{\chi_{-y^m}(X)}{m}q^m\right)
\end{aligned}
$$

(22) follows easily from (28).

$$
\begin{aligned}
&\sum_{n\geq 0}\operatorname{sign}(X^{(n)}q^n = \exp\left(\sum_{m>1}\frac{\chi_{-(-1)^m}(X)}{m}q^m\right) \\
= \quad & \exp\left(\sum_{l>1}\frac{\chi_{-(-1)^{2l}}(X)}{2l}q^{2l}\right)\cdot\exp\left(\sum_{l>1}\frac{\chi_{-(-1)^{2l-1}}(X)}{2l-1}q^{2l-1}\right) \\
= \quad & \exp\left(\sum_{l>1}\frac{\chi(X)}{2l}q^{2l}\right)\cdot\exp\left(\sum_{l>1}\frac{\operatorname{sign}(X)}{2l-1}q^{2l-1}\right) \\
= \quad & \frac{1}{(1-q^2)^{\chi(X)/2}}\left(\frac{1+q}{1-q}\right)^{\operatorname{sign}(X)/2}.
\end{aligned}
$$

□

For a Riemann surface X, the formulas in Theorem 5.1 and Corollary 5.1 match with the results of Macdonald [21]. Hirzebruch has computed the generating functional of the signature of symmetric products of any compact oriented manifold (for details, see Zagier [28], §9). Formula (23) matches with his result for complex manifolds.

Given a complex orbifold M/G, M^g is a complex submanifold for every $g \in M$. One can then consider the delocalized equivariant Dolbeault cohomology

$$H^{*,*}(M,G) = \bigoplus_{[g]\in G_*} H^{*,*}(M^g)^{Z_g}.$$

It follows from (15) that

$$H^{*,*}(X^n,S_n) \cong \bigoplus_{\sum lN_l=n} H^{*,*}(\prod_{l=1}^{n} X^{(N_l)}) = \bigoplus_{\sum lN_l=n}\bigotimes_{l=1}^{n} H^{*,*}(X^{(N_l)}).$$

Using the isomorphism $H^{*,*}(M/G) \cong H^{*,*}(M)^G$, we get

$$H^{*,*}(X^n,S_n) \cong \bigoplus_{\sum lN_l=n}\bigotimes_{l=1}^{n} S^{N_l}(H^{*,*}(X))$$

as vector spaces. Let $\dim_{\mathbb{C}} X = 2k$ for some $k \in \frac{1}{2}\mathbb{Z}$, by Example 2.1, as $\frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$-graded vector spaces,

$$H^{*,*}(X^n, S_n) \cong \bigoplus_{\sum lN_l = n} \bigotimes_{l=1}^{n} S^{N_l}(H^{*,*}(X)[k(l-1), k(l-1)]). \tag{24}$$

Of course, when M is even dimensional, both sided are $\mathbb{Z} \times \mathbb{Z}$-graded. By (13), we have

$$\begin{aligned}
& \sum_{n \geq 0} h_{x,y}(X^n, S_n) q^n \\
= & \sum_{n \geq 0} q^n \sum_{\sum lN_l = n} h_{x,y}(\bigotimes_{l=1}^{n} S^{N_l}(H^{*,*}(X)[k(l-1), k(l-1)])) \\
= & \sum_{n \geq 0} q^n \sum_{\sum lN_l = n} \prod_{l \geq 1} h_{x,y}(S^{N_l}(H^{*,*}(X)[k(l-1), k(l-1)])) \\
= & \prod_{l \geq 1} \sum_{N_l \geq 0} h_{x,y}(S^{N_l}(H^{*}(X)[k(l-1), k(l-1)])) q^{lN_l} \\
= & \prod_{l \geq 1} \frac{\prod_{s+t\ odd}(1 + x^s y^t q^l)^{h^{s-k(l-1), t-k(l-1)}(X)}}{\prod_{s+t\ even}(1 - x^s y^t q^l)^{h^{s-k(l-1), t-k(l-1)}(X)}} \\
= & \prod_{l \geq 1} \frac{\prod_{s+t\ odd}(1 + x^{s+k(l-1)} y^{t+k(l-1)} q^l)^{h^{s,t}(X)}}{\prod_{s+t\ even}(1 - x^{s+k(l-1)} y^{t+k(l-1)} q^l)^{h^{s,t}(X)}}.
\end{aligned}$$

To summarize, we have proved the following

Theorem 5.2. *For a closed complex manifold X of dimension $2k$ for some $k \in \frac{1}{2}\mathbb{Z}$, if $H^{*,*}(X^n, S_n)$ is $\frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$-graded as in (17), then we have*

$$\sum_{n \geq 0} h_{x,y}(X^n, S_n) q^n = \prod_{l \geq 1} \frac{\prod_{s+t\ odd}(1 + x^{s+k(l-1)} y^{t+k(l-1)} q^l)^{h^{s,t}(X)}}{\prod_{s+t\ even}(1 - x^{s+k(l-1)} y^{t+k(l-1)} q^l)^{h^{s,t}(X)}}. \tag{25}$$

For a complex orbifold M/G, it is natural to define

$$\chi_y(M, G) = \sum_{s,t \geq 0} (-1)^t h^{s,t}(M, G) y^s.$$

Clearly we have $\chi_{-1}(M, G) = \chi(M, G)$. It is natural to regard $\chi_1(M, G)$ as the signature of (M, G). Similarly, we regard $p_a(M, G) = \chi_0(M, G)$ as the arithmetic genus of (M, G). Similar to $\chi(M, G)$, $\chi_y(M, G)$ is a sum of contributions from M^g, $[g] \in G_*$. Denote by $h^{r,s}(M^g)^{Z_g}$ the dimension of $H^{r,s}(M^g)^{Z_g} \cong H^{r,s}(M^g/Z_g)$. Since

$$H^{p,q}(M, G) = \bigoplus_{[g] \in G_*} H^{p-F_g, q-F_g}(M^g)^{Z_g} = \bigoplus_{[g] \in G_*} H^{p-F_g, q-F_g}(M^g/Z_g),$$

we have

$$\begin{aligned}
\chi_y(M, G) &= \sum_{[g] \in G_*} \sum_{r,s \geq 0} (-1)^{t+F_g} h^{r,s}(M^g)^{Z_g} y^{s+F_g} \\
&= \sum_{[g] \in G_*} \sum_{r,s \geq 0} (-1)^{t+F_g} h^{r,s}(M^g/Z_g) y^{s+F_g}.
\end{aligned}$$

Therefore,

$$\chi_y(M,G) = \sum_{[g]\in G_*} (-y)^{F_g}\chi_y(M^g)^{Z_g} = \sum_{[g]\in G_*} (-y)^{F_g}\chi_y(M^g/Z_g), \tag{26}$$

where

$$\begin{aligned}\chi_y(M^g)^{Z_g} &= \sum_{s,t\geq 0}(-1)^t h^{s,t}(M^g)^{Z_g}y^s,\\ \chi_y(M^g/Z_g) &= \sum_{s,t\geq 0}(-1)^t h^{s,t}(M^g/Z_g)y^s.\end{aligned}$$

Taking $y=1$, we get

$$\chi_1(M,G) = \sum_{[g]\in G_*} (-1)^{F_g}\chi_y(M^g)^{Z_g} = \sum_{[g]\in G_*} (-1)^{F_g}\chi_y(M^g/Z_g). \tag{27}$$

This indicates how one should define the orbifold signature.

Corollary 5.2. *Under the assumptions of Theorem 5.2, we have*

$$\sum_{n\geq 0}\chi_{-y}(X^n,S_n)q^n = \exp\left(\sum_{n>0}\frac{q^n}{n}\frac{\chi_{-y^n}(X)}{1-(y^kq)^n}\right), \tag{28}$$

$$\sum_{n\geq 0} p_a(X^n,S_n)q^n = \frac{1}{(1-q)^{p_a(X)}}. \tag{29}$$

Proof. From (25) we get

$$\begin{aligned}&\sum_{n\geq 0}\chi_{-y}(X^n,S_n)q^n = \prod_{l\geq 1}\frac{\prod_{s+t\ odd}(1+(-y)^{s+k(l-1)}(-1)^{t+k(l-1)}q^l)^{h^{s,t}(X)}}{\prod_{s+t\ even}(1-(-y)^{s+k(l-1)}(-1)^{t+k(l-1)}q^l)^{h^{s,t}(X)}}\\ = &\prod_{l\geq 1}\frac{\prod_{s+t\ odd}(1-y^{s+k(l-1)}q^l)^{h^{s,t}(X)}}{\prod_{s+t\ even}(1-y^{s+k(l-1)}q^l)^{h^{s,t}(X)}}.\end{aligned}$$

The logarithm of the last term is

$$\begin{aligned}&-\sum_{l\geq 1}\sum_{s,t\geq 0}(-1)^{s+t}h^{s,t}(X)\log(1-y^{s+k(l-1)}q^l)\\ =&\sum_{l\geq 1}\sum_{s,t\geq 0}(-1)^{s+t}h^{s,t}(X)\sum_{n>0}\frac{1}{n}(y^{s+k(l-1)}q^l)^n\\ =&\sum_{n>0}\frac{1}{n}\sum_{l\geq 1}\sum_{s,t\geq 0}(-1)^{s+t}h^{s,t}(X)y^{sn+kn(l-1)}q^{ln}\\ =&\sum_{n>0}\frac{1}{n}\chi_{-y^n}(X)\sum_{l\geq 1}y^{kn(l-1)}q^{ln} = \sum_{n>0}\frac{q^n}{n}\frac{\chi_{-y^n}(X)}{1-(y^kq)^n}.\end{aligned}$$

When $x=0$, $x^{s+k(l-1)}\neq 0$ only if $s=0$ and $l=1$, hence

$$\sum_{n\geq 0}p_a(X^n,S_n)q^n = \frac{\prod_{t\ odd}(1+(-1)^tq)^{h^{0,t}(X)}}{\prod_{t\ even}(1-(-1)^tq)^{h^{0,t}(X)}} = \frac{1}{(1-q)^{p_a(X)}}.$$

□

We omit the elementary proof of the following formulas:

$$\exp\left(\sum_{l\geq 1}\frac{q^{2l}}{2l}\frac{1}{1-q^{2l}}\right)=\prod_{m\geq 1}\frac{1}{(1-q^{2m})^{1/2}},$$

$$\exp\left(\sum_{l\geq 1}\frac{q^{2l-1}}{2l-1}\frac{1}{1-q^{2l-1}}\right)=\prod_{m\geq 1}\left(\frac{1+q^m}{1-q^m}\right)^{1/2},$$

$$\exp\left(\sum_{l\geq 1}\frac{q^{2l-1}}{2l-1}\frac{1}{1+q^{2l-1}}\right)=\prod_{m\geq 1}\left(\frac{1+q^m}{1-q^m}\right)^{(-1)^{m+1}/2}.$$

Corollary 5.3. *Under the assumptions of Theorem 5.2, we have*

$$\sum_{n\geq 0}\chi_1(X^n,S_n)q^n=\prod_{m\geq 1}\left(\frac{1}{(1-q^{2m})^{\chi(X)/2}}\left(\frac{1+q^m}{1-q^m}\right)^{(-1)^{k(m+1)}\operatorname{sign}(X)/2}\right).$$

Proof. We take $y=-1$ in (28). For k even, we have

$$\begin{aligned}\sum_{n\geq 0}\chi_1(X^n,S_n)q^n &= \exp\left(\sum_{n>0}\frac{q^n}{n}\frac{\chi_{-(-1)^n}(X)}{1-q^n}\right)\\ &= \exp\left(\sum_{l\geq 1}\frac{q^{2l}}{2l}\frac{\chi_{-1}(X)}{1-q^{2l}}\right)\cdot\exp\left(\sum_{l\geq 1}\frac{q^{2l-1}}{2l-1}\frac{\chi_1(X)}{1-q^{2l-1}}\right)\\ &= \prod_{m\geq 1}\left(\frac{1}{(1-q^{2m})^{\chi(X)/2}}\left(\frac{1+q^m}{1-q^m}\right)^{\operatorname{sign}(X)/2}\right).\end{aligned}$$

For k odd, we have

$$\begin{aligned}\sum_{n\geq 0}\chi_1(X^n,S_n)q^n &= \exp\left(\sum_{n>0}\frac{(-1)^n}{n}\frac{\chi_{-(-1)^n}(X)}{1-(-q)^n}\right)\\ &= \exp\left(\sum_{l\geq 1}\frac{q^{2l}}{2l}\frac{\chi_{-1}(X)}{1-q^{2l}}\right)\cdot\exp\left(\sum_{l\geq 1}\frac{q^{2l-1}}{2l-1}\frac{\chi_1(X)}{1+q^{2l-1}}\right)\\ &= \prod_{m\geq 1}\left(\frac{1}{(1-q^{2m})^{\chi(X)/2}}\left(\frac{1+q^m}{1-q^m}\right)^{(-1)^{m+1}\operatorname{sign}(X)/2}\right).\end{aligned}$$

□

6. The B-version

In the study of mirror symmetry of Calabi-Yau manifolds, one encounters another interesting Dolbeault cohomology algebra of a closed complex manifold M:

$$H^{-*,*}(M)=\bigoplus_{p,q\geq 0}H^q(M,\Lambda^p(TM))=\bigoplus_{p,q\geq 0}H^{-p,q}(M),$$

with the algebra structure induced from the wedge products. We will refer to it as the *B-algebra*. Clearly all the results in last section have a version for $H^{-*,*}$, so we will be brief.

Set $h^{-p,q}(M) = \dim H^{-p,q}(M)$. The B-Hodge polynomial is by definition:

$$\hat{h}_{x,y}(M) = \sum_{p,q\geq 0} h^{-p,q}(M) x^p y^q.$$

For a Calabi-Yau d-fold M, Serre duality shows that $h^{-p,q}(M) = h^{d-p,q}$. Hence we have

$$\begin{aligned}\hat{\chi}_{-y}(M) &= \sum_{p,q\geq 0} (-1)^q h^{d-p,q}(M)(-y)^p \\ &= \sum_{p,q\geq 0} (-1)^q h^{p,q}(M)(-y)^{d-p} = (-y)^d \chi_{-y^{-1}}(M).\end{aligned}$$

Or equivalently

$$y^{d/2}\hat{\chi}_{-y}(M) = (-1)^{d/2} \cdot y^{-d/2}\chi_{-y^{-1}}(M). \tag{30}$$

For a complex orbifold M/G,

$$H^{-*,*}(M/G) = H^{-*,*}(M)^G.$$

One define $\hat{h}_{x,y}(M/G)$ in a similar fashion. In particular,

$$H^{*,*}(X^{(n)}) \cong S^n(H^{*,*}(X)),$$

for any closed complex manifold X. One then has the following analogue of Theorem 5.1:

Theorem 6.1. *For a compact complex manifold X, we have*

$$\sum_{n\geq 0} \hat{h}_{x,y}(X^{(n)}) q^n = \frac{\prod_{s+t\ odd}(1 + x^s y^t q)^{h^{-s,t}(X)}}{\prod_{s+t\ even}(1 - x^s y^t q)^{h^{-s,t}(X)}}. \tag{31}$$

The B-version of Hirzebruch's χ_y-genus for a compact complex manifold M is defined to be

$$\hat{\chi}_y(M) = \sum_{p,q\geq 0} (-1)^q h^{-p,q}(M) y^p = \hat{h}_{y,-1}(M).$$

Corollary 6.1. *For a compact complex manifold X we have*

$$\sum_{n\geq 0} \hat{\chi}_{-y}(X^{(n)}) q^n = \exp\left(\sum_{m\geq 1} \frac{\hat{\chi}_{-y^m}(X)}{m} q^m\right). \tag{32}$$

Given a complex orbifold M/G, consider the B-delocalized equivariant Dolbeault cohomology

$$H^{-*,*}(M,G) = \bigoplus_{[g]\in G_*} H^{-*,*}(M^g)^{Z_g}.$$

We give it a bigrading by fractional numbers by

$$H^{p,q}(M,G) = \bigoplus_{[g]\in G_*} H^{p-F_g,q-F_g}(M^g)^{Z_g} = \bigoplus_{[g]\in G_*} H^{p-F_g,q-F_g}(M^g/Z_g).$$

In particular, it follows from (15) and Example 2.1 that

$$H^{-*,*}(X^n, S_n) \cong \bigoplus_{\sum lN_l = n} \bigotimes_{l=1}^{n} S^{N_l}(H^{-*,*}(X)[k(l-1), k(l-1)]). \tag{33}$$

where $\dim_{\mathbb{C}} X = 2k$ for some $k \in \frac{1}{2}\mathbb{Z}$. Again, when M is even dimensional, both sided are $\mathbb{Z} \times \mathbb{Z}$-graded.

Theorem 6.2. *For a closed complex manifold X of dimension $2k$ for some $k \in \frac{1}{2}\mathbb{Z}$, if $H^{*,*}(X^n, S_n)$ is $\frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$-graded as in (33), then we have*

$$\sum_{n\geq 0} \hat{h}_{x,y}(X^n, S_n)q^n = \prod_{l\geq 1} \frac{\prod_{s+t\ odd}(1 + x^{s+k(l-1)}y^{t+k(l-1)}q^l)^{h^{-s,t}(X)}}{\prod_{s+t\ even}(1 - x^{s+k(l-1)}y^{t+k(l-1)}q^l)^{h^{-s,t}(X)}}. \tag{34}$$

Define

$$\hat{\chi}_y(M, G) = \sum_{s,t\geq 0} (-1)^t h^{-s,t}(M, G)y^s.$$

Then we have

Corollary 6.2. *Under the assumptions of Theorem 6.2, we have*

$$\sum_{n\geq 0} \hat{\chi}_{-y}(X^n, S_n)q^n = \exp\left(\sum_{n>0} \frac{q^n}{n} \frac{\hat{\chi}_{-y^n}(X)}{1 - (y^k q)^n}\right). \tag{35}$$

Similar to $\chi(M, G)$ and $\chi_y(M, G)$, we have

$$\hat{\chi}_y(M, G) = \sum_{[g]\in G_*} (-y)^{F_g} \hat{\chi}_y(M^g/Z_g). \tag{36}$$

7. A special case of a formula for elliptic genera of symmetric products

Our results above give a mathematical proof of a special case of a formula for the elliptic genera of symmetric products found by Dijkgraaf, Moore, E. Verlinde and H. Verlinde [8]. For a closed complex d-manifold M, expand

$$E_{q,y} = y^{-\frac{d}{2}} \bigotimes_{n\geq 1} (\Lambda_{-yq^{n-1}} TM \otimes \Lambda_{-y^{-1}q^n} T^*M \otimes S_{q^n} TM \otimes S_{q^n} T^*M)$$

as a sum of holomorphic vector bundles:

$$E_{q,y} = \bigoplus_{m\geq 0,l} q^m y^l E_{m.l}.$$

It is easy to see that each $E_{l,m}$ is of finite rank. Let $c(m, l)$ be the Riemann-Roch number of $E_{m,l}$, then the elliptic genus defined in [8] is

$$\chi(M; q, y) = \sum_{m\geq 0,l} c(m, l)q^m y^l.$$

It was also described in [8] as the trace of some operator on some Hilbert space $\mathcal{H}(M)$, but the exact description of this Hilbert space and the operators involved were not explicitly given. Furthermore, for a complex orbifold M/G, the definition of its orbifold elliptic genus used such a description: the total orbifold Hilbert space takes the form

$$\mathcal{H}(M, G) = \oplus_{[g]\in G_*} \mathcal{H}_g^{Z_g},$$

where $\mathcal{H}_g$ is a Hilbert space with a Z_g action on it. String theoretical arguments used in [8] requires the Z_g-action satisfies some nice properties in the case of symmetric products, and the following formula was derived

$$\sum_{n\geq 0} p^n \chi(X^n, S_n; q, y) = \prod_{n>0,m\geq 0,l} \frac{1}{(1 - p^n q^m y^l)^{c(nm,l)}}, \tag{37}$$

where $\chi(X^n, S_n; q, y)$ denotes the orbifold elliptic genus of the symmetric product, $\chi(X; q, y) = \sum c(m, l) q^m y^l$.

Mathematically, the definition of the orbifold elliptic genus has yet to be worked out. But the following discussion provides some hint on the right definition. Presumably, it involves the contributions from the fixed point sets M^g for $[g]$ running through all conjugacy classes. We look at two special cases. When $y = 1$, Dijkgraaf *et al* [8] claimed that $\chi(M; q, 1) = \chi(M)$, the Euler number of M. The orbifold Euler number as shown by Hirzebruch and Höfer [17] is given by formula (7) in §1:

$$\chi(M, G) = \sum_{[g] \in G_*} \chi(M^g / Z_g).$$

When $q = 0$, Dijkgraaf *et al* [8] claimed that

$$\chi(M; 0, y) = \sum_{r,s} (-1)^{r+s} y^{r - \frac{d}{2}} h^{r,s}(M) = y^{-\frac{d}{2}} \chi_{-y}(M).$$

(It seems to the author that $\chi(M; 0, y)$ should be equal to $y^{-\frac{d}{2}} \hat{\chi}_{-y}(M)$, since $E_{0,y} = y^{-\frac{d}{2}} \Lambda_{-y} TM$.) Formula (37) reduces to

$$\sum_{n \geq 0} p^n \chi(X^n, S_n; 0, y) = \prod_{n > 0, l \geq 0} \frac{1}{(1 - p^n y^l)^{c(0,l)}}. \tag{38}$$

We now rewrite the right hand side as follows:

$$\begin{aligned} \prod_{n>0, l \geq 0} \frac{1}{(1 - p^n y^l)^{c(0,l)}} &= \exp\left(\sum_{n>0} \sum_{l \geq 0} -c(0, l) \log(1 - p^n y^l) \right) \\ = \quad \exp\left(\sum_{n>0} \sum_{l \geq 0} c(0, l) \sum_{m>0} \frac{(p^n y^l)^m}{m} \right) &= \exp\left(\sum_{n>0} \sum_{m>0} \chi(X; 0, y^m) \frac{p^{nm}}{m} \right) \\ = \quad \exp\left(\sum_{m>0} \frac{\chi(X; 0, y^m)}{m} \frac{p^m}{1 - p^m} \right). & \end{aligned}$$

So (38) is equivalent to

$$\sum_{n \geq 0} p^n \chi(X^n, S_n; 0, y) = \exp\left(\sum_{m>0} \frac{\chi(X; 0, y^m)}{m} \frac{p^m}{1 - p^m} \right). \tag{39}$$

Noting the similarity between (39) and (28), we are led to the following:

Theorem 7.1. *If one defines*

$$\chi(M, G; 0, y) = y^{-\frac{\dim M}{2}} \chi_{-y}(M, G) \tag{40}$$

for complex orbifolds M/G, then (39) holds for any closed complex manifold X.

Proof. We use (28) for $q = y^{-k}p$, where $\dim_{\mathbb{C}} = 2k$ for some $k \in \frac{1}{2}\mathbb{Z}$:

$$\begin{aligned} \sum_{n\geq 0} p^n \chi(X^n, S_n; 0, y) &= \sum_{n\geq 0} \chi_y(X^n, S_n) y^{-kn} p^n \\ &= \exp\left(\sum_{m>0} \frac{(y^{-k}p)^m}{m} \frac{\chi_{-y^m}(X)}{1-p^m} \right) \\ &= \exp\left(\sum_{m>0} \frac{(y^m)^{-k}\chi_{-y^m}(X)}{m} \frac{p^m}{1-p^m} \right) \\ &= \exp\left(\sum_{m>0} \frac{\chi(X; 0, y^m)}{m} \frac{p^m}{1-p^m} \right). \end{aligned}$$

□

By (26), (40) becomes

$$\begin{aligned} \chi(M, G; 0, y) &= y^{-\frac{\dim M}{2}} \sum_{[g]\in G_*} y^{F_g} \chi_{-y}(M^g/Z_g) \\ &= \sum_{[g]\in G_*} y^{-\frac{\dim M}{2} + F_g + \frac{\dim M^g}{2}} \chi(M^g/Z_g; 0, y). \end{aligned}$$

In the case of symmetric products, $F_g = \frac{\dim M - \dim M^g}{2}$, hence if we use (40), then

$$\chi(X^n, S_n; 0, y) = \sum_{[g]\in(S_n)_*} \chi((X^n)^g/Z_g; 0, y).$$

It seems reasonable to conjecture that if one defines

$$\chi(X^n, S_n; q, y) = \sum_{[g]\in(S_n)_*} \chi((X^n)^g/Z_g; q, y),$$

then (37) holds.

We also have the B-version of the above results:

Theorem 7.2. *If one defines*

$$\hat{\chi}(M, G; 0, y) = y^{-\frac{\dim M}{2}} \hat{\chi}_{-y}(M, G)$$

for complex orbifolds M/G, then

$$\hat{\chi}(X^n, S_n; 0, y) = \sum_{[g]\in(S_n)_*} \hat{\chi}((X^n)^g/Z_g; 0, y).$$

Furthermore, (39) holds with χ replaced by $\hat{\chi}$ for any closed complex manifold X.

8. Heisenberg superalgebra representation

Set $\mathcal{F}(X) = \sum_{n\geq 0} H^*(X^n, S_n)$. Vafa and Witten [26] noticed that this space admits an action by an infinite dimensional Heisenberg superalgebra. Following Segal [25] and Wang [27], we give a construction of such an action in terms of the some functors defined in §2. We first recall some well-known facts about Heisenberg superalgebras (see e.g. Kac [18]). Let $\mathfrak{h}$ be a finite dimensional complex superspace with a non-degenerate supersymmetric bilinear form $\eta(\cdot, \cdot)$. In other words, $\mathfrak{h}$ is

$\mathbb{Z}_2$-graded: $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$, $\eta : \mathfrak{h} \otimes \mathfrak{h} \to \mathbb{C}$ is a nondegenerate bilinar functional of degree $\bar{0}$, such that

$$\eta(a,b) = (-1)^{\bar{a}\cdot\bar{b}}\eta(b,a),$$

for homogeneous $a, b \in \mathfrak{h}$. Here we have used $\bar{a}$ to denote the degree of a. We will refer to η simply as a pairing on $\mathfrak{h}$. Consider the Lie superalgebra (Kac [18], §3.5)

$$\hat{\mathfrak{h}} = \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{h} \oplus \mathbb{C}K$$

with commutation relations

$$[a_m, b_n] = m \cdot \eta(a,b)\delta_{m,-n}K, \quad [K, \hat{\mathfrak{h}}] = 0,$$

where $m, n \in \mathbb{Z}, a, b \in \mathfrak{h}$, $a_m = t^m a$ and $b_n = t^n b$. Clearly $\hat{\mathfrak{h}}$ is the direct sum of $\mathfrak{h}$ with Heisenberg superalgebra $\hat{\mathfrak{h}}'$ given by

$$\hat{\mathfrak{h}}' = \hat{\mathfrak{h}}^{<} \oplus \mathbb{C}K \oplus \hat{\mathfrak{h}}^{>},$$

where $\hat{h}^{<} = \oplus_{n\geq 1}(t^{-n}\mathfrak{h})$ and $\hat{\mathfrak{h}}^{>} = \oplus_{n\geq 1}(t^{n}\mathfrak{h})$. Let $\hat{\mathfrak{h}}^{+} = \hat{\mathfrak{h}}^{>} \oplus \mathbb{C}K$. Given $k \in \mathbb{C}$, denote by π^k the 1-dimensional representation of $\hat{\mathfrak{h}}^{+}$ defined by

$$\pi^k(\hat{\mathfrak{h}}) = 0, \quad \pi(K) = k.$$

The Verma module $\widetilde{V}^k(\mathfrak{h}, \eta) = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}^{+})} \pi^k$ has an explicit description as follows:

$$\widetilde{V}^k(\mathfrak{h}, \eta) = S^*(\hat{\mathfrak{h}}^{<}) = S^*(\oplus_{l\geq 1} t^{-l}\mathfrak{h}) = \otimes_{l\geq 1} S^*(t^{-l}\mathfrak{h}), \tag{41}$$

where K acts as multiplication by a constant k, for $m > 0$, a_{-m} acts by multiplication, a_m by km times the contraction:

$$(t^m \otimes a) \cdot (t^{-n} \otimes b) = km\delta_{m,n}\eta(a,b).$$

When $k \neq 0$, there is a structure of a simple conformal vertex algebra on $\widetilde{V}^k$ (Kac [18], §4.7 and §4.10). It is called the *free bosonic vertex algebra*. This vertex operator algebra structure is essentially determined by the action of $\hat{\mathfrak{h}}'$ on $\widetilde{V}^k(\mathfrak{h}, \eta)$.

It is straightforward to generalize to the $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$-graded version. For a $\mathbb{Z}$-graded $\mathfrak{h}$, we denote by $|a|$ the degree of a homogeneous element $a \in \mathfrak{h}$. We assume that the η is a nondegenerate graded symmetric bilinear form of degree 0, we can give t an arbitrary even degree $-d$. Then K acts as an operator of degree 0, and for $m > 0$, a_{-m} acts by an operator of degree $|a| + md$, a_m acts by an operator of degree $-|a| - md$. Similarly, for a $\mathbb{Z} \times \mathbb{Z}$-graded $\mathfrak{h}$, we denote by $||a||$ the bi-degree of a homogeneous element $a \in \mathfrak{h}$. We assume that η is a nondegenerate bilinear form of bi-degree $(0,0)$, bigraded symmetric in the sense that

$$\eta(a,b) = (-1)^{(p_1+q_1)(p_2,q_2)}\eta(b,a)$$

for $a \in \mathfrak{h}_{p_1.q_1}$, $b \in \mathfrak{h}_{p_2,q_2}$. We give t a bi-degree (p,p). Then K acts as an operator of bi-degree $(0,0)$, and for $m > 0$, $a \in \mathfrak{h}_{r,s}$ a_{-m} acts as an operator of bidegree $(r + mp, t + mp)$, a_m acts as an operator of bidegree $(-(r + mp), -(t + mp))$.

From the isomorphism (17), we get

$$\begin{aligned}
\mathcal{F}(X) &\cong \sum_{n\geq 0} \bigoplus_{\sum lN_l = n} \bigotimes_{l=1}^{n} S^{N_l}(H^*(X)[d(l-1)]) \\
&= \bigotimes_{l\geq 1} \sum_{N_l\geq 0} S^{N_l}(H^*(X)[d(l-1)]) = \bigotimes_{l\geq 1} S^*(H^*(X)[d(l-1)]) \\
&= S^*(\oplus_{l\geq 1} H^*(X)[d(l-1)]).
\end{aligned}$$

Comparing with (41), we see that when d is an even number, if t has degree $-d$, then we have
$$\sum_{n\geq 0} H^*(X^n, S_n) \cong \widetilde{V}^k(H^*(X)[-d], \eta),$$
where $\eta(\alpha, \beta) = \int_X \alpha \cup \beta$ for $\alpha, \beta \in H^*(X)$. Here we have assumed that X is a connected closed oriented manifold of dimension $2d$ with d even, hence by Poincaré duality, η is a nondegenerate graded symmetric bilinear form on $H^*(X)[-d]$.

It follows from the above discussion that $\mathcal{F}(X)$ is the Verma module of the Heisenberg superalgebra for $(H^*(X)[-d], \eta)$. Now we give geometric realization of the relevant operators. We first recall some maps defined in §2. For any positive integers $p < n$, the inclusion $S_p \times S_{n-p} \hookrightarrow S_n$ induces two maps
$$\begin{aligned} \mathrm{Res} &: H^*(X^n, S_n) \to H^*(X^n, S_p \times S_{n-p}), \\ \mathrm{Ind} &: H^*(X^n, S_p \times S_{n-p}) \to H^*(X^n, S_n). \end{aligned}$$
We also have the Künneth isomorphism
$$\kappa : H^*(X^n, S_p \times S_{n-p}) \to H^*(X^p, S_p) \otimes H^*(X^{n-p}, S_{n-p}).$$
Clearly $\mathrm{Res}, \mathrm{Ind}$ and κ all have degree 0 with the given $\mathbb{Z}$-grading on relevant spaces. For $0 \leq m \leq n$, define
$$\cdot : H^r(X^m, S_m) \otimes H^*(X^{n-m}, S_{n-m}) \to H^*(X^n, S_n)$$
as the composition
$$H^*(X^m, S_m) \otimes H^*(X^{n-m}, S_{n-m}) \xrightarrow{\kappa^{-1}} H^*(X^n, S_m \times S_{n-m}) \xrightarrow{\mathrm{Ind}} H^*(X^n, S_n),$$
also define
$$\Delta : H^*(X^n, S_n) \to \oplus_{m=0}^n H^*(X^m, S_m) \otimes H^*(X^{n-m}, S_{n-m})$$
as the direct sum of the compositions
$$H^*(X^n, S_n) \xrightarrow{\mathrm{Res}} H^*(X^n, S_m \times S_{n-m}) \xrightarrow{\kappa} H^*(X^m, S_m) \otimes H^*(X^{n-m}, S_{n-m}).$$
Hence we get maps $\cdot : \mathcal{F}(X) \otimes \mathcal{F}(X) \to \mathcal{F}(X)$ and $\Delta : \mathcal{F}(X) \to \mathcal{F}(X) \otimes \mathcal{F}(X)$. It is easy to verify that under the identification, $\mathcal{F}(X) \cong S^*(\oplus_{l\geq 1} t^l H^*(X)[-m])$, $\cdot$ and Δ gives the multiplication and comultiplication for the standard Hopf algebra structure on the latter space. For an n-cycles $\sigma_n \in S_n$, $Z_{\sigma_n} = \langle \sigma_n \rangle$ and $H^*((X^n)^{\sigma_n})^{Z_{\sigma_n}} \cong H^*(X)$. So there are two homomorphisms
$$\begin{aligned} i_n &: H^*(X) \to H^*(X^n, S_n), \\ \pi_n &: H^*(X^n, S_n) \to H^*(X) \end{aligned}$$
induced by the inclusion and projection respectively. When regarded as a map from $H^*(X)[-d]$ to $H^*(X^n, S_n)$, i_n has degree nd; similarly, π_n is a map of degree $-nd$ from $H^*(X^n, S_n)$ to $H^*(X)[-d]$. For any $1 \leq m \leq n$ and $\alpha \in H^*(X)$, define
$$p_m(\alpha) : H^*(X^n, S_n) \to H^*(X^{n-m})$$
as m times the composition
$$\begin{aligned} H^*(X^n, S_n) &\xrightarrow{\Delta} H^*(X^m, S_m) \otimes H^*(X^{n-m}, S_{n-m}) \\ &\xrightarrow{\pi_m \otimes 1} H^*(X) \otimes H^*(X^{n-m}, S_{n-m}) \xrightarrow{\iota_\alpha \otimes 1} H^*(X^{n-m}, S_{n-m}). \end{aligned}$$
Also define
$$q_m(\alpha) : H^*(X^{n-m}) \to H^*(X^n, S_n)$$

as the composition

$$\begin{aligned} H^*(X^{n-m}, S_{n-m}) &\xrightarrow{\alpha\otimes 1} H^*(X) \otimes H^*(X^{n-m}, S_{n-m}) \\ &\xrightarrow{i_m\otimes 1} H^*(X^m, S_m) \otimes H^*(X^{n-m}, S_{n-m}) \xrightarrow{\cdot} H^*(X^n, S_n) \end{aligned}$$

Then $p_m(\alpha)$ has degree $-(|\alpha| - d) - md$, and $q_m(\alpha)$ has degree $(|\alpha| - d) + md$. It is straightforward to check that $p_m(\alpha)$ is m times the contraction by $t^m \otimes \alpha$ and $q_m(\alpha)$ is the multiplication by $t^{-m} \otimes \alpha$. For an open manifold X, one can use the natural pairing on the direct sum of $H_c^*(X)$ with its dual space to construct a Heisenberg superalgebra and its action on $\mathcal{F}_c(X) = \sum_{n\geq 0} H_c^*(X^n, S_n)$.

When M is a closed complex manifold, one can also consider the space

$$\widehat{\mathcal{F}}(X) = \sum_{n\geq 0} H^{-*,*}(X^n, S_n).$$

When X is Calabi-Yau d-fold, there is a pairing $\hat{\eta}$ given by the composition:

$$H^{-p,q}(X) \otimes H^{-r,s}(X) \xrightarrow{\wedge} H^{-(p+r),q+s}(X) \xrightarrow{\sharp} H^{d-(p+r),q+s}(X) \xrightarrow{\int_X} \mathbb{C},$$

where $\sharp$ is the isomorphism induced by the holomorphic volume form on X. Similar to the above discussion, one gets an action of the Heisenberg superalgebra constructed from $(H^{-*,*}(X), \eta)$ on $\widehat{\mathcal{F}}(X)$. In general, one use the natural pairing on $H^{-*,*}(M)$ and its dual space to construct a Heisenberg superalgebra and its representation on $\widehat{\mathcal{F}}(X)$.

9. Surface case

When X is a smooth algebraic surface, the Hilbert scheme $X^{[n]}$ of 0-dimensional subscheme of length n is a smooth algebraic variety. Göttsche [13] has shown that

$$\sum_{n\geq 0} P_t(X^{[n]})q^n = \prod_{l\geq 1} \frac{(1+t^{2l-1}q^l)^{b_1(X)}(1+t^{2l+1}q^l)^{b_3(X)}}{(1-t^{2l-2}q^l)^{b_0(X)}(1-t^{2l}q^l)^{b_2(X)}(1-t^{2l+2}q^l)^{b_4(X)}}.$$

Also Göttsche-Soergel [15] and Cheah [7] have shown by different methods that

$$\sum_{n\geq 0} h_{x,y}(X^{[n]})q^n = \prod_{l\geq 1} \frac{\prod_{s+t\ odd}(1+x^{s+(l-1)}y^{t+(l-1)}q^l)^{h^{s,t}(X)}}{\prod_{s+t\ even}(1-x^{s+(l-1)}y^{t+(l-1)}q^l)^{h^{s,t}(X)}}.$$

Compared with our formulas for $H^*(X^n, S_n)$ and $H^{*,*}(X^n, S_n)$, we see that for a smooth algebraic surface X,

$$H^*(X^{[n]}) \cong H^*(X^n, S_n), \tag{42}$$

$$H^{*,*}(X^{[n]}) \cong H^{*,*}(X^n, S_n) \tag{43}$$

as (bi-)graded vector spaces. As a consequence, $\chi(X^{(n)}) = \chi(X^n, S_n)$ for smooth algebraic surfaces. This was noticed by Hirzebruch and Höfer [17]. In [15], the grading shift actually has an explanation in terms of intersection cohomology.

The isomorphisms (42) and (43) are special cases of a very general result. As was proved in a paper of Batyrev and Dais [5], $h^{p,q}(M, G) = h^{p,q}(\widehat{M/G})$ for a crepant, full desingularization $\widehat{M/G} \to M/G$ of the orbifold M/G, whenever the so-called "strong McKay correspondence" holds for all quotient singularities occurring along every stratum of X/G. For a smooth algebraic surface, the Hilbert scheme $X^{[n]}$ provides a crepant resolution of $X^{(n)}$. Göttsche [14] showed that this resolution satisfies the "strong McKay correspondence", while Batyrev proved it in general in

his recent preprint [4]. If one uses such results, our calculations give a method of computing the Hodge numbers of $X^{[n]}$.

It seems to be a general phenomenon that invariants of $X^{[n]}$ can be computed by suitably defined corresponding invariants of $X^{(n)}$ for surfaces. If one can relate the construction of Nakajima [22] and Grojnowski [16] for the Hilbert schemes to our construction in §7 or the constructions in Segal [25] and Wang [27], then one obtains another example of this phenomenon. Also related is a conjecture in Dijkgraaf-Moore-Verlinde-Verlinde [8] on the elliptic genera for the Hilbert schemes of K3 surfaces. Liu [19] has proposed a method to directly verify it. Nevertheless the study of the relationship with the elliptic genera of the symmetric products still seems interesting.

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